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Forced convection from a heated flat plate

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Heat-transfer coefficients are calculated for forced convection from a heated flat plate, of finite breadth and infinite span, at zero incidence to a steady stream of viscous, incompressible fluid. The complete range of the Reynolds number R is considered and the results for large R are compared with the Pohlhausen boundary-layer solution for a plate of infinite breadth with the Blasius velocity field.

It is found that the heat transfer from the trailing edge of the plate is important for small Reynolds numbers but steadily diminishes as R increases. The limiting value of the heat-transfer coefficient at the leading edge agrees to good accuracy with Pohlhausen's result and the corresponding overall heat-transfer coefficient is within 3 % of Pohlhausen's value for an equal length of the infinite plate measured from the leading edge. The results over the whole Reynolds-number range are probably correct to this order of accuracy.

The Reynolds analogy between skin friction and heat transfer, exactly true at large Reynolds numbers, is found to be inadequate at small values of R, as may then be expected, due to the existence of a pressure gradient parallel to the plate.

Introduction

The problem considered is that of finding the theoretical temperature field $T(x,y)$ in a viscous, incompressible fluid in steady two-dimensional motion past a flat plate of finite breadth $2c$, whose cross-section occupies the position $y=0$, $-c \leqslant x \leqslant c$, where (x,y) are Cartesian co-ordinates. The fluid properties are assumed to be independent of temperature, and heat dissipation within the fluid is neglected. If forced convection only is assumed, the governing equation is

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=\frac{\rho c_p}{k}\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right). \tag{1}$$

Here ρ, c_p and k are respectively the density, specific heat at constant pressure, and the coefficient of thermal conductivity of the fluid; also (u,v) are the components of velocity. The plate is assumed to be maintained at surface temperature T_1; and at large distances from it, where the flow reduces to a uniform stream $u=U$, $v=0$, the temperature is assumed to be uniform and equal to T_0.

For two-dimensional motion a stream function exists. For the finite flat plate at zero incidence, Dennis & Dunwoody (1966) have shown how to calculate the dimensionless stream function $\psi(=1/Uc$ times the dimensional stream function) in the form

$$\psi(\xi,\eta) = \sum_{n=1}^{\infty} f_n(\xi)\sin n\eta, \tag{2}$$

where (ξ,η) are elliptic co-ordinates defined by the transformation

$$x = c\cosh\xi\cos\eta, \quad y = c\sinh\xi\sin\eta. \tag{3}$$

This transformation maps the upper half of the xy-plane (which by symmetry is all that need be considered) into the semi-infinite strip $\xi \geqslant 0,\ 0 \leqslant \eta \leqslant \pi$. The plate transforms to $\xi = 0$, with leading edge at $\eta = \pi$ and trailing edge at $\eta = 0$. In terms of the dimensionless stream function ψ and the temperature function

$$\theta = (T-T_0)/(T_1-T_0), \tag{4}$$

equation (1) becomes

$$\nabla^2\theta = \tfrac{1}{2}R\sigma\left(\frac{\partial\psi}{\partial\eta}\frac{\partial\theta}{\partial\xi}-\frac{\partial\psi}{\partial\xi}\frac{\partial\theta}{\partial\eta}\right), \tag{5}$$

where $\nabla^2 = \partial^2/\partial\xi^2+\partial^2/\partial\eta^2$. Here

$$R = 2cU/\nu, \quad \sigma = \rho\nu c_p/k$$

are the Reynolds and Prandtl numbers, respectively, ν being the coefficient of kinematical viscosity. The boundary conditions are that

$$\left.\begin{aligned}\theta &= 1, \quad \text{when} \quad \xi = 0;\\ \theta &\to 0, \quad \text{as} \quad \xi \to \infty.\end{aligned}\right\} \tag{6}$$

A boundary-layer solution to the forced convection problem has been given by Pohlhausen (1921) for a plate of infinite breadth at high Reynolds numbers. The finite plate has not previously been considered at any Reynolds number, so that, for example, no previous results are available on the effect of the trailing edge. In the present paper the stream function calculated by Dennis & Dunwoody in the form (2) over the range $R = 0{\cdot}1$ to ∞ is used to obtain solutions of the heat-transfer equation (5). The variation of heat transfer along the plate is calculated, and, in particular, results for the heat-transfer coefficients α_L and α_T at the leading and trailing edges are presented in the form

$$\left.\begin{aligned}\alpha_L &= kA(R,\sigma)\sqrt{(U/\nu X)},\\ \alpha_T &= kB(R,\sigma)\sqrt{\{U/\nu(2c-X)\}},\end{aligned}\right\} \tag{7}$$

where X is distance measured along the plate from the leading edge. The mean Nusselt number N is expressed as

$$N = C(R,\sigma)R^{\frac{1}{2}}. \tag{8}$$

The coefficients A, B and C in these formulae are tabulated in detail as functions of R for $\sigma = 0{\cdot}73$ (air) and $\sigma = 1$. It is found that, to reasonably numerical accuracy,

$$C(R,\sigma) \approx C(R,1)\,\sigma^{\frac{1}{3}} \tag{9}$$

for any R and σ within the range considered. It can be shown directly from the governing equations that (9) is asymptotically correct for fixed R and large enough σ. Moreover, as $R \to \infty$, $C(R,1) \to C$, an absolute constant. From the numerical results its value is found to be 0·685, about 3 % higher than Pohlhausen's value of 0·664. Similar results to (9) hold for $A(R,\sigma)$ and $B(R,\sigma)$; and as $R\to\infty$ it is found that $A(R,1)\to 0{\cdot}332$, which agrees with Pohlhausen's value.

Method of analysis

Apart from the trivial change of replacing a factor R by $R\sigma$, equation (5) for $\theta(\xi,\eta)$ is identical with the equation governing the scalar vorticity $\zeta(\xi,\eta)$ in the problem of steady flow past a flat plate. The essential difference in the problems is that θ and ζ satisfy different boundary conditions, in particular, ζ is an odd function of η in the range $\eta=-\pi$ to $\eta=\pi$, while θ must clearly be an even function. The method used to reduce the equation for ζ to a set of ordinary differential equations in the independent variable ξ may therefore be adapted to the present problem by putting

$$\theta=\phi\exp\{F(\xi,\eta)\},$$

where F is a solution of the equation

$$\partial F/\partial\xi=\tfrac{1}{4}R\sigma\,\partial\psi/\partial\eta,$$

and expressing ϕ in the form

$$\phi(\xi,\eta)=g_0(\xi)+2\sum_{n=1}^{\infty}g_n(\xi)\cos n\eta. \tag{10}$$

Hence $g_n(\xi)\,(n=0,1,2,\ldots)$ is effectively the Fourier cosine transform of ϕ (whereas in the flow problem it is the sine transform) and the reduction of (5) to a set of linear differential equations

$$g_n''-n^2g_n+\sum_{p=0}^{\infty}k_{n,p}\,g_p=0\quad(n=0,1,2,\ldots) \tag{11}$$

follows similar steps to those set out by Dennis & Dunwoody. In terms of the same four functions $a_n(\xi)$, $b_n(\xi)$, $c_n(\xi)$ and $d_n(\xi)$, all derived from $f_n(\xi)$, defined in the paper cited, it may be verified that

$$\delta_p k_{n,p}(\xi)=\tfrac{1}{8}R\sigma\{(n-p)\,c_{n+p}+(n+p)\,c_{n-p}\}-\frac{R^2\sigma^2}{64}\sum_{r=1}^{\infty}(M_r+N_r), \tag{12}$$

where $\delta_0=2;\ \delta_p=1\quad(p=1,2,3,\ldots).$

In this formula $M_r=d_r(d_{n-p-r}+d_{n-p+r}+d_{n+p-r}+d_{n+p+r})$ (13)

and $N_r=a_r(b_{r+n-p}+b_{r-n+p}+b_{r-n-p}+b_{r+n+p}).$ (14)

Negative suffixes are interpreted according to definitions already given (*loc. cit*).

Numerical solutions of (11) which satisfy the conditions (6) are obtained using methods described for the flow problem. A particular solution of the system (11) is obtained by assuming numerical boundary conditions at a sufficiently large value ξ_0 of ξ. For $\xi>\xi_0$ an approximate solution of (11) (asymptotically correct as $\xi\to\infty$) is obtained using the method of Jeffreys & Jeffreys (1962). This approximate solution automatically satisfies the second condition of (6) and it is then joined at $\xi=\xi_0$ to an inner solution computed using finite-difference methods. Independent numerical solutions of this form, denoted for fixed p by the aggregate solution

$$\{g_n^{(p)}(\xi)\} \quad (n = 0, 1, 2, \ldots),$$

are obtained by making the boundary conditions at $\xi = \xi_0$ distinct for each value of p. These can then be combined into a complete solution of (11) which satisfies the second condition of (6) by writing

$$g_n(\xi) = \sum_{p=0}^{\infty} C_p g_n^{(p)}(\xi) \quad (n = 0, 1, 2, \ldots). \tag{15}$$

The constants C_p are then found to satisfy the first condition of (6). Since the function $F(\xi, \eta)$ has been chosen to satisfy the condition

$$F(0, \eta) = 0,$$

the first of (6) requires that

$$\phi(0, \eta) = 1$$

and hence that

$$g_0(0) = 1; \quad g_n(0) = 0 \quad (n = 1, 2, 3, \ldots).$$

If the series (15) is now substituted, the constants C_p must satisfy the simultaneous equations

$$\left.\begin{aligned} \sum_{p=0}^{\infty} C_p g_n^{(p)}(0) &= 1 \quad (n = 0) \\ &= 0 \quad (n = 1, 2, 3, \ldots). \end{aligned}\right\} \tag{16}$$

The $g_n^{(p)}(0)$ are known from the numerical solutions. Hence the constants may be found and the solution for ϕ arrived at in the form (10), all the conditions of the problem being satisfied.

Calculated results

Heat-transfer coefficients at the plate are calculated from the value of $\partial\theta/\partial\xi$ at $\xi = 0$. This quantity is denoted, for convenience, by $G(\eta)$. Since

$$F = \partial F/\partial\xi = 0 \quad \text{when} \quad \xi = 0,$$

then

$$G(\eta) = g_0'(0) + 2\sum_{n=1}^{\infty} g_n'(0)\cos n\eta. \tag{17}$$

The numerical calculations were limited to five terms of the series. In tables 1 and 2 values of the first five coefficients of the series (17) are given for $\sigma = 0{\cdot}73$ and $\sigma = 1$, respectively, over a wide range of Reynolds numbers. The tabulated values correspond to the coefficients

$$h_n = -\pi R^{-\frac{1}{2}} g_n'(0). \tag{18}$$

It is clear from the tables that the coefficients h_n tend to definite limits as $R \to \infty$; and it will now be shown that the definition of h_n leads to the result

$$h_0 = C(R, \sigma). \tag{19}$$

The quantity of heat transferred from the plate to the fluid per unit area and time at co-ordinate η is

$$q(\eta) = -k\left(\frac{\partial T}{\partial y}\right)_{y=0} = \frac{-k}{c\sin\eta}\left(\frac{\partial T}{\partial\xi}\right)_{\xi=0},$$

from the transformation (3). If a heat-transfer coefficient $\alpha(\eta)$ is defined by the equation

$$q(\eta) = (T_1 - T_0)\,\alpha(\eta)$$

then

$$\alpha(\eta) = -kG(\eta)/c \sin \eta. \tag{20}$$

R	A	B	h_0	h_1	h_2	h_3	h_4
0·1	0·661	0·626	2·022	−0·028	—	—	—
0·2	0·548	0·493	1·636	−0·043	—	—	—
0·4	0·459	0·380	1·319	−0·062	−0·001	—	—
1	0·395	0·281	1·064	−0·089	−0·001	—	—
2	0·367	0·223	0·932	−0·113	−0·002	—	—
4	0·352	0·176	0·840	−0·138	−0·005	—	—
10	0·344	0·132	0·769	−0·166	−0·011	−0·001	—
20	0·337	0·101	0·724	−0·183	−0·017	−0·003	−0·001
40	0·329	0·077	0·692	−0·192	−0·024	−0·006	−0·003
100	0·319	0·063	0.675	−0·191	−0·030	−0·010	−0·008
500	0·314	0·057	0·664	−0·190	−0·030	−0·012	−0·010
1,000	0·312	0·055	0·658	−0·189	−0·029	−0·013	−0·012
10,000	0·303	0·047	0·631	−0·187	−0·026	−0·014	−0·014
∞	0·299	0·045	0·616	−0·185	−0·023	−0·014	−0·015

TABLE 1. $\sigma = 0{\cdot}73$; details of the numerical solution

R	A	B	h_0	h_1	h_2	h_3	h_4
0·1	0·709	0·661	2·152	−0·037	—	—	—
0·2	0·593	0·521	1·750	−0·056	—	—	—
0·4	0·499	0·398	1·411	−0·079	−0·001	—	—
1	0·436	0·294	1·151	−0·111	−0·002	—	—
2	0·409	0·234	1·018	−0·137	−0·004	—	—
4	0·394	0·184	0·925	−0·164	−0·008	−0·001	—
10	0·386	0·137	0·852	−0·193	−0·015	−0·003	—
20	0·377	0·104	0·804	−0·210	−0·022	−0·005	−0·002
40	0·370	0·082	0·773	−0·218	−0·028	−0·008	−0·004
100	0·358	0·067	0·755	−0·216	−0·034	−0·013	−0·010
500	0·350	0·061	0·738	−0·213	−0·033	−0·014	−0·013
1,000	0·347	0·059	0·731	−0·212	−0·032	−0·014	−0·014
10,000	0·337	0·052	0·700	−0·209	−0·029	−0·015	−0·016
∞	0·332	0·048	0·685	−0·207	−0·026	−0·016	−0·018

TABLE 2. $\sigma = 1$; details of the numerical solution

The total rate of heat transfer per unit width from both sides of the plate is

$$Q = 2\int_0^{2c} q\,dX,$$

and the mean Nusselt number N is then defined by the equation

$$Q = 2kN(T_1 - T_0).$$

Hence

$$N = -\int_0^{\pi} G(\eta)\,d\eta = -\pi g_0'(0). \tag{21}$$

Comparing this result with (8) establishes (19); and the limit of $C(R, 1)$ as $R \to \infty$ is seen from table 2 to be 0·685.

The heat-transfer coefficients α_L and α_T at the leading and trailing edges are found by considering the limiting forms of (20) at $\eta = \pi$ and $\eta = 0$, respectively. First, near $\eta = \pi$,

$$\sin\eta = \sin(\pi - \eta') \sim \eta',$$

while from (3)

$$x + c = X \sim \tfrac{1}{2}c\eta'^2.$$

Hence at $\eta = \pi$

$$\alpha_L = -k(2cX)^{-\frac{1}{2}}G(\pi).$$

Comparing this with the first of (7) gives

$$A(R,\sigma) = -G(\pi)R^{-\frac{1}{2}}. \tag{22}$$

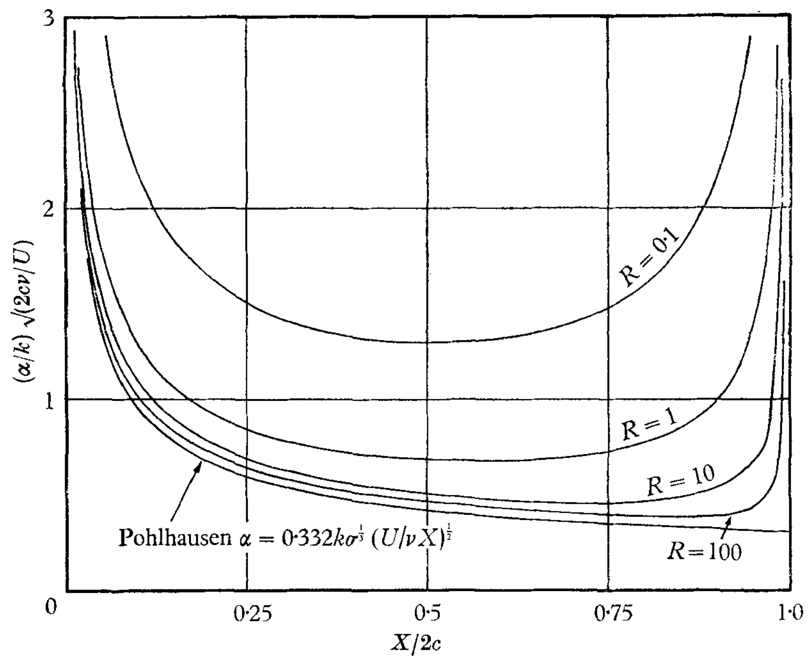

FIGURE 1

Similarly, by expanding for small η near $\eta = 0$, the result

$$B(R,\sigma) = -G(0)R^{-\frac{1}{2}} \tag{23}$$

is obtained. These two coefficients are given in tables 1 and 2 for the respective cases $\sigma = 0{\cdot}73$ and $\sigma = 1$.

For small Reynolds numbers it is seen that α_T is of considerable importance; in fact, as $R \to 0$

$$\alpha_L \approx \alpha_T, \tag{24}$$

a consequence of the fact that the function $g_0(\xi)$ then dominates the series (10). On the other hand, as $R \to \infty$, α_T diminishes. The limiting value of $A(R, 1)$ as $R \to \infty$ is 0·332. To this number of figures this agrees exactly with Pohlhausen's boundary-layer value. The variation of heat transfer along the length of the plate is shown, for various Reynolds numbers, in figure 1 for the case $\sigma = 0{\cdot}73$ (air). This has been calculated directly from (20) using the results of table 1. The approach to the Pohlhausen solution for large R is clearly seen.

Calculations have also been carried out over a range of values of the Prandtl number σ. Some results for the mean Nusselt number N are given in table 3. From these it will be seen that the variation of $N\sigma^{-\frac{1}{3}}$ is small for all Reynolds numbers and that, for fixed R, this quantity approaches a definite limit as σ becomes large. Moreover, the limit is approached more rapidly as R becomes large and for $R > 20$ it has been found that the variation of $N\sigma^{-\frac{1}{3}}$ is insignificant for all values of σ in the range considered.

R \ σ	0·73	1	8	64	512	4,096	32,768
0·2	0·813	0·783	0·601	0·537	0·518	0·515	0·515
0·4	0·927	0·893	0·744	0·690	0·677	0·677	—
1	1·182	1·151	1·028	0·990	0·984	0·984	—
2	1·464	1·439	1·345	1·320	1·320	—	—
4	1·866	1·850	1·790	1·780	1·780	—	—
10	2·700	2·694	2·682	2·682	—	—	—
20	3·596	3·594	3·593	—	—	—	—

TABLE 3. Values of $N\sigma^{-\frac{1}{3}}$

These properties follow on account of the well-known fact that a thermal boundary layer exists whose thickness, for fixed R, decreases with increasing σ. This is experienced during the numerical calculations and can, in fact, be demonstrated directly from the equations (11). Suppose that a value $\xi = \delta_1$ (not to be confused with δ_p in equation (12)) denotes the thermal boundary-layer thickness. The functions $k_{n,p}(\xi)$ in (11) depend upon $f_n(\xi)$, which satisfy the initial conditions

$$f_n(0) = f'_n(0) = 0 \quad (n = 1, 2, 3, \ldots).$$

Assuming that $f_n(\xi)$ can be expanded as Taylor series for $\xi \leqslant \delta_1$, we may write

$$f_n(\xi) = (1/2!)\,\xi^2 f''_n(0)\{1 + O(\delta_1)\} \tag{25}$$

within the thermal boundary layer, where $f''_n(0)$ depends upon R alone. The terms in $k_{n,p}(\xi)$, equation (12), are of two types, depending respectively on $R\sigma$ and $R^2\sigma^2$. If substitution for the $f_n(\xi)$ in the form (25) is made then, as $\delta_1 \to 0$,

$$k_{n,p}(\xi) = R\sigma\xi\{\lambda_{n,p} + O(\delta_1)\} + R^2\sigma^2\xi^4\{\mu_{n,p} + O(\delta_1)\}, \tag{26}$$

where the constants $\lambda_{n,p}$ and $\mu_{n,p}$ are, for fixed n and p, functions of R alone.

The variable ξ is now changed to the new variable t by the relation $\xi = \delta_1 t$, and equations (11) become

$$\frac{d^2 g_n}{dt^2} - \delta_1^2 n^2 g_n + \delta_1^2 \sum_{p=0}^{\infty} k_{n,p} g_p = 0. \tag{27}$$

From (26), as $\delta_1 \to 0$,

$$\delta_1^2 k_{n,p} \sim R\sigma\delta_1^3\lambda_{n,p} t + R^2\sigma^2\delta_1^6\mu_{n,p} t^4.$$

If δ_1 is now chosen such that $R\sigma\delta_1^3 = \text{const.}$ then, for fixed R, $\delta_1 = O(\sigma^{-\frac{1}{3}})$ and the functions $\delta_1^2 k_{n,p}$ tend to functions of t alone, i.e. independent of σ but depending of course, on R. We may omit the term in n^2 in (27) which, for fixed n, tends to zero with δ_1. Solutions for the g_n can then be obtained which are functions of t

alone. These determine values of dg_n/dt which are independent of σ and hence the limiting value of the mean Nusselt number $N(R, \sigma)$ is

$$N(R, \infty) = -\pi\delta_1^{-1}(dg_0/dt)_0 = O(\sigma^{\frac{1}{3}})$$

as $\sigma \to \infty$. A similar result holds for the local heat-transfer coefficient.

A limiting solution to the problem may also be considered when, for fixed σ, R becomes large. There is then a boundary-layer thickness $\xi = \delta$ associated with the velocity field. If we put $\xi = \delta z$, equations (11) become

$$\frac{d^2g_n}{dz^2} - \delta^2 n^2 g_n + \delta^2 \sum_{p=0}^{\infty} k_{n,p} g_p = 0. \tag{28}$$

These equations may now be considered in the same manner as that previously described by Dennis & Dunwoody for the corresponding set of equations in the determination of the velocity field. For it is apparent that the $k_{n\ p}(\xi)$ defined by (12) depend upon the functions $f_n(\xi)$ in the same manner (although their aggregate values are different) as the corresponding quantities in the flow problem. Also the Reynolds number occurs explicitly in the same way. Since, as R becomes large, $\delta = O(R^{-\frac{1}{2}})$, it may be shown, as in the flow problem, that

$$\delta^2 k_{n,p}(\xi) \sim K_{n,p}(z),$$

i.e. functions of z alone, independent of R but depending on σ in the present case. The limiting functions $K_{n,p}(z)$ are calculated in terms of z from the same formulae (12) but with certain modifications in the definitions of the quantities b_n and c_n already noted in the paper cited and, further, with R replaced by the numerical constant c' which represents the grouping $R\delta^3 B_1$, where B_1 is defined in the paper cited and is $O(R^{\frac{1}{2}})$ as R becomes large.

For fixed n the term in n^2 in (28) tends to zero with δ, and limiting solutions for the g_n can be found as functions of z alone. The limiting value of the mean Nusselt number for large R is then

$$N(\infty, \sigma) = -\pi\delta^{-1}(dg_0/dz)_0 = O(R^{\frac{1}{2}}),$$

with a similar result for the local coefficient.

If R and σ are both large it follows that

$$N \sim CR^{\frac{1}{2}}\sigma^{\frac{1}{3}}. \tag{29}$$

This can also be deduced directly from the limiting solutions of the equations (27). It follows from the high Reynolds-number results for the flow problem given by Dennis & Dunwoody that, as $R \to \infty$, the quantities $\lambda_{n,p}$ and $\mu_{n,p}$ in (26) are of the order of $R^{\frac{1}{2}}$ and R, respectively. Hence if R and σ are both large, $\delta_1^2 k_{n,p}$ in (27) can be made independent of both R and σ by choosing δ_1 such that

$$R^{\frac{3}{2}}\sigma\delta_1^3 = \text{const.},$$

the limiting solutions of (27) then being independent of both R and σ, leading at once to (29). It may be noted, however, that it is not possible to deduce that (29) holds for large R and *any* σ since, when R becomes large, we cannot represent each $f_n(\xi)$ by the leading term in its Taylor series unless, in addition, $\delta_1 \to 0$

independently of the behaviour of R, i.e. unless $\sigma \to \infty$ also. Nevertheless, the numerical results show that N is proportional to $\sigma^{\frac{1}{3}}$ for quite small σ if R is large enough. For all practical purposes the variation of the local coefficient of heat transfer with σ is directly proportional to the corresponding variation of the mean coefficient, so that a complete set of results for all R and σ may be deduced from the results of tables 1, 2 and 3.

h_0	h_1	h_2	h_3	h_4
0·664	−0·221	−0·044	−0·019	−0·011

TABLE 4. Fourier coefficients according to Pohlhausen's solution

The limiting results $R = \infty$ in tables 1 and 2 have been calculated by solving equations (28) using the velocity field in terms of the co-ordinate z calculated by Dennis & Dunwoody. A detailed comparison between these results and the Pohlhausen solution may be made in the following way. When $\sigma = 1$ the Pohlhausen solution gives the local heat-transfer coefficient as

$$\alpha(\eta) = 0{\cdot}332k\,(U/\nu X)^{\frac{1}{2}},$$

with $$X = c(1+\cos\eta).$$

By (20), this leads to the result

$$G(\eta) = -0{\cdot}332R^{\frac{1}{2}}\sin\tfrac{1}{2}\eta$$

and hence from (17) and (18)

$$h_n = 0{\cdot}664/(1-4n^2).$$

The first five of these coefficients are shown in table 4. The differences between them and the corresponding coefficients of table 2 for the case $R = \infty$ are not great when measured as a percentage of the mean coefficient h_0. It seems, however, that although the present calculations determine the heat-transfer coefficient at the leading edge accurately for large R, truncation of the series (17) leads to an erroneous estimate of the trailing-edge heat transfer. This is indicated by the fact that the coefficient $B(R, \sigma)$ does not vanish when $R = \infty$, whereas it seems likely that in theory it should do so.

The equation for the velocity component $u(x, y)$ parallel to the plate is

$$-\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}. \tag{30}$$

It is well known (Schlichting 1960) that when the pressure gradient $\partial p/\partial x = 0$ in (30) and $\sigma = 1$ in (1) the two equations are analogous and the functions u/U and $(T-T_1)/(T_0-T_1)$, which satisfy similar differential equations with the same boundary conditions, are identical. This leads to the Reynolds analogy

$$q = \{k(T_1-T_0)/\rho\nu U\}\,p_{xy}$$

between the heat transfer q and the skin friction p_{xy}, and hence by integration over the whole plate to

$$N = \tfrac{1}{2}RC_D, \tag{31}$$

where C_D is the drag coefficient. Using values of C_D calculated by Dennis & Dunwoody, some values of the ratio $RC_D/2N$ are shown in table 5. The accuracy to which (31) is satisfied for $R = \infty$ verifies, within the limits of the numerical methods, that the calculated velocity field for large R is associated with a zero pressure gradient. As R decreases, the effect of the pressure gradient increases and (31) becomes correspondingly less valid.

R	0·1	1	10	100	1000	∞
$RC_D/2N$	1·665	1·581	1·388	1·245	1·086	1·014

TABLE 5. The Reynolds analogy between the total drag coefficient and the mean heat transfer coefficient for $\sigma = 1$

Solution at low Reynolds numbers

For low Reynolds numbers a solution of (5) may be obtained, following the method of Oseen, by taking the stream function to be that for the external potential flow past the plate, viz.

$$\psi = \sinh\xi\sin\eta.$$

Substituting in (5) and putting

$$\theta = \chi\exp\left(\tfrac{1}{4}R\sigma\cosh\xi\cos\eta\right)$$

leads to the equation

$$\nabla^2\chi - 2k^2(\cosh 2\xi - \cos 2\eta)\,\chi = 0, \tag{32}$$

with $k = \frac{1}{8}R\sigma$. Separation of the variables in (32) yields, as is well known, two ordinary differential equations of Mathieu and modified Mathieu types respectively (McLachlan 1947) and a solution for χ which is an even function of η and which ensures that $\theta \to 0$ as $\xi \to \infty$ is found to be

$$\chi = \sum_{n=0}^{\infty}\beta_n Fek_n(\xi, -k^2)\,ce_n(\eta, -k^2).$$

Here the notation of McLachlan is adopted and the β_n are arbitrary constants. These have to be determined to satisfy the first of the conditions (6). Once this has been done we have

$$\left(\frac{\partial\theta}{\partial\xi}\right)_{\xi=0} = e^{\frac{1}{4}R\sigma\cos\eta}\sum_{n=0}^{\infty}\beta_n Fek_n'(0, -k^2)\,ce_n(\eta, -k^2),$$

and the terms in the series for $G(\eta)$ may be evaluated by equating coefficients of $\cos n\eta$ $(n = 0, 1, 2, \ldots)$. The whole process is tedious and only the final results are quoted. It is found that

$$g_0'(0) \approx \frac{1}{S} + \left(2+\frac{1}{S}\right)k^2 - \left(2S - \frac{5}{4} - \frac{3}{16S}\right)k^4 \tag{33}$$

correct to terms in k^4, while

$$g_n'(0) = O(k^n) \quad (n \neq 0). \tag{34}$$

Here $S = \gamma + \log \frac{1}{2}k$, where γ is Euler's constant. Equations (33) and (34) together tend to confirm the result (24), in agreement with the previous calculated results. Some values of the mean Nusselt number calculated from (33) in the case $\sigma = 1$ are compared with the main results of the paper in table 6. These indicate that the Oseen solution is adequate only at very low Reynolds numbers.

R	Calculated	Oseen	R	Calculated	Oseen
0·1	0·681	0·698	1	1·151	1·351
0·2	0·783	0·822	2	1·440	1·779
0·4	0·892	0·996	4	1·850	2·765

TABLE 6. Comparison of N for $\sigma = 1$

REFERENCES

DENNIS, S. C. R. & DUNWOODY, J. 1966 *J. Fluid Mech.* **24**, 577.

JEFFREYS, H. & JEFFREYS, B. S. 1962 *Methods of Mathematical Physics*, 3rd ed., p. 522. Cambridge University Press.

McLACHLAN, N. W. 1947 *Theory and Application of Mathieu Functions*. Oxford: Clarendon Press.

POHLHAUSEN, E. 1921 *ZAMM*, **1**, 115.

SCHLICHTING, H. 1960 *Boundary Layer Theory*, 4th ed. p. 303. New York: McGraw-Hill.